The following is an excerpt from Chapter 10 of *Calculus, Early Transcendentals* by James Stewart. It has been reformatted slightly from its appearance in the textbook so that it corresponds to the LaTeX article style.

1 The Binomial Series

You may be acquainted with the Binomial Theorem, which states that if a and b are any real numbers and k is a positive integer, then

$$\begin{aligned}(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 \\ + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}a^{k-n}b^n \\ + \cdots + kab^{k-1} + b^k\end{aligned}$$

The traditional notation for the binomial coefficient is

$$\binom{k}{0} = 1 \qquad \binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \qquad n = 1, 2, \ldots, k$$

which enables us to write the Binomial Theorem in the abbreviated form

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$$

In particular, if we put $a = 1$ and $b = x$, we get

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n \tag{1}$$

One of Newton's accomplishments was to extend the Binomial Theorem (Equation 1) to the case where k is no longer a positive integer. In this case the expression for $(1+x)^k$ is no longer a finite sum; it becomes an infinite series.

Assuming that $(1+x)^k$ can be expanded as a power series, we compute its Maclaurin series in the usual way:

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

If u_n is the nth term of this series, then

$$\begin{aligned} \left| \frac{u_{n+1}}{u_n} \right| &= \left| \frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n} \right| \\ &= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| \to |x| \qquad \text{as } n \to \infty \end{aligned}$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

Definition 1 (The Binomial Series). If k is any real number and $|x| < 1$, then

$$\begin{aligned} (1+x)^k &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \end{aligned}$$

where

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad (n \geq 1) \qquad \binom{k}{0} = 1$$

We have proved Definition 1 under the assumption that $(1+x)^k$ has a power series expansion. For a proof without that assumption see Exercise 21.

Although the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, ± 1, depends on the value of k. It turns out that the series converges at 1 if $-1 < k \leq 0$ and at both endpoints if $k \geq 0$. Notice that if k is a positive integer and $n > k$, then the expression for $\binom{k}{n}$ contains a factor $(k-k)$, so $\binom{k}{n} = 0$ for $n > k$. This means that the series terminates and reduces to the ordinary Binomial Theorem (Equation 1) when k is a positive integer.

Although, as we have seen, the binomial series is just a special case of the Maclaurin series, it occurs frequently and so it is worth remembering.

Example 1. Expand $\frac{1}{(1+x)^2}$ as a power series.

Solution. We use the binomial series with $k = -2$. The binomial coefficient is

$$\begin{aligned} \binom{-2}{n} &= \frac{(-2)(-3)(-4)\cdots(-2-n+1)}{n!} \\ &= \frac{(-1)^n 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n(n+1)}{n!} = (-1)^n(n+1) \end{aligned}$$

and so, when $|x| < 1$,

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n$$
$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \qquad \square$$

Example 2. Find the Maclaurin series for the function $f(x) = 1/\sqrt{4-x}$ and its radius of convergence.

Solution. As given, $f(x)$ is not quite of the form $(1+x)^k$ so we rewrite it as follows:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Using the binomial series with $k = -\frac{1}{2}$ and with x replaced by $-x/4$, we have

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n$$
$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3\right.$$
$$\left. + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^n + \cdots\right]$$
$$= \frac{1}{2}\left(1 + \frac{1}{8}x + \frac{1\cdot 3}{2!\,8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!\,8^3}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,8^n}x^n + \cdots\right)$$

We know from Definition 1 that this series converges when $|-x/4| < 1$, that is, $|x| < 4$, so the radius of convergence is $R = 4$. $\square$