1 The Binomial Series

You may be acquainted with the Binomial Theorem, which states that if $a$ and $b$ are any real numbers and $k$ is a positive integer, then

\[
(a + b)^k = a^k + k a^{k-1} b + \frac{k(k-1)}{2!} a^{k-2} b^2 + \frac{k(k-1)(k-2)}{3!} a^{k-3} b^3 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} a^{k-n} b^n + \cdots + k a b^{k-1} + b^k
\]

The traditional notation for the binomial coefficient is

\[
\binom{k}{0} = 1, \quad \binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}, \quad n = 1, 2, \ldots, k
\]

which enables us to write the Binomial Theorem in the abbreviated form

\[
(a + b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n
\]

In particular, if we put $a = 1$ and $b = x$, we get

\[
(1 + x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n
\]  

(1)

One of Newton's accomplishments was to extend the Binomial Theorem (Equation 1) to the case where $k$ is no longer a positive integer. In this case the expression for $(1 + x)^k$ is no longer a finite sum; it becomes an infinite series.

Assuming that $(1 + x)^k$ can be expanded as a power series, we compute its Maclaurin series in the usual way:

\[
\begin{align*}
f(x) &= (1 + x)^k & f(0) &= 1 \\
f'(x) &= k(1 + x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1 + x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1 + x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&
\vdots
\\f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1 + x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{align*}
\]
\[(1 + x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n\]

If \(u_n\) is the \(n\)th term of this series, then

\[
\frac{|u_{n+1}|}{u_n} = \frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n} = \frac{|k-n|}{n+1} |x| = \frac{1 - \frac{k}{n+1}}{1 + \frac{1}{n}} |x| \rightarrow |x| \quad \text{as } n \rightarrow \infty
\]

Thus, by the Ratio Test, the binomial series converges if \(|x| < 1\) and diverges if \(|x| > 1\).

**Definition 1 (The Binomial Series).** If \(k\) is any real number and \(|x| < 1\), then

\[
(1 + x)^k = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots
= \sum_{n=0}^{\infty} \binom{k}{n} x^n
\]

where

\[
\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad (n \geq 1) \quad \binom{k}{0} = 1
\]

We have proved Definition 1 under the assumption that \((1 + x)^k\) has a power series expansion. For a proof without that assumption see Exercise 21.

Although the binomial series always converges when \(|x| < 1\), the question of whether or not it converges at the endpoints, \(\pm 1\), depends on the value of \(k\). It turns out that the series converges at 1 if \(-1 < k \leq 0\) and at both endpoints if \(k \geq 0\). Notice that if \(k\) is a positive integer and \(n > k\), then the expression for \(\binom{k}{n}\) contains a factor \((k-k)\), so \(\binom{k}{n} = 0\) for \(n > k\). This means that the series terminates and reduces to the ordinary Binomial Theorem (Equation 1) when \(k\) is a positive integer.

Although, as we have seen, the binomial series is just a special case of the Maclaurin series, it occurs frequently and so it is worth remembering.

**Example 1.** Expand \(\frac{1}{(1+x)^2}\) as a power series.

**Solution.** We use the binomial series with \(k = -2\). The binomial coefficient is

\[
\binom{-2}{n} = \frac{(-2)(-3)(-4)\cdots(-2-n+1)}{n!} = \frac{(-1)^n 2 \cdot 3 \cdot 4 \cdots n(n+1)}{n!} = (-1)^n(n+1)
\]
and so, when \(|x| < 1\),
\[
\frac{1}{(1+x)^2} = (1+x)^{-2} = \sum_{n=0}^{\infty} \left(-\frac{2}{n}\right)x^n
\]
\[
= \sum_{n=0}^{\infty} (-1)^n(n+1)x^n
\]

\[\Box\]

**Example 2.** Find the Maclaurin series for the function \(f(x) = 1/\sqrt{4-x}\) and its radius of convergence.

**Solution.** As given, \(f(x)\) is not quite of the form \((1+x)^k\) so we rewrite it as follows:
\[
\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4(1-\frac{x}{4})}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-1/2}
\]

Using the binomial series with \(k = -\frac{1}{2}\) and with \(x\) replaced by \(-x/4\), we have
\[
\frac{1}{\sqrt{4-x}} = \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-1/2} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{5}{2}\right) \cdots \left(-\frac{n}{2}\right) \left(-\frac{x}{4}\right)^n
\]
\[
= \frac{1}{2} \left[ 1 + \left(-\frac{1}{2}\right) \left(-\frac{x}{4}\right) + \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{x}{4}\right)^2 + \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{5}{2}\right) \cdots \left(-\frac{n}{2}\right) \left(-\frac{x}{4}\right)^n + \cdots \right]
\]
\[
= \frac{1}{2} \left( 1 + \frac{1}{8}x + \frac{1\cdot 3}{2! 8^2}x^2 + \frac{1\cdot 3\cdot 5}{3! 8^3}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdots (2n-1)}{n! 8^n}x^n + \cdots \right)
\]

We know from Definition 1 that this series converges when \(|-x/4| < 1\), that is, \(|x| < 4\), so the radius of convergence is \(R = 4\).

\[\Box\]